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LETTER TO THE EDITOR

Sobolev, Hardy and CLR inequalities associated with Pauli operators in $\mathbb{R}^3$

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Abstract

In a previous article, the first two authors have proved that the existence of zero modes of Pauli operators is a rare phenomenon; *inter alia*, it is proved that for $|\boldsymbol{B}| \in L^{3/2}(\mathbb{R}^3)$, the set of magnetic fields $\boldsymbol{B}$ which do not yield zero modes contains an open dense subset of $[L^{3/2}(\mathbb{R}^3)]^3$. Here the analysis is taken further, and it is shown that Sobolev, Hardy and Cwikel–Lieb–Rosenbljum (CLR) inequalities hold for Pauli operators for all magnetic fields in the aforementioned open dense set.

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1. Introduction

The Pauli operator is formally defined by

$$\mathbb{P}_A = \left\{\boldsymbol{\sigma}\cdot\left(\frac{1}{\mathrm{i}}\boldsymbol{\nabla} + \boldsymbol{A}\right)\right\}^2 \equiv \sum_{j=1}^{3}\left\{\sigma_j\left(\frac{1}{\mathrm{i}}\partial_j + A_j\right)\right\}^2 \tag{1}$$

where $\boldsymbol{A} = (A_1, A_2, A_3)$ is a vector potential which is such that $\mathrm{curl}\boldsymbol{A} = \boldsymbol{B}$, the magnetic field, and $\boldsymbol{\sigma} \equiv (\sigma_1, \sigma_2, \sigma_3)$ is the triple of Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2}$$

Expression (1) defines a non-negative self-adjoint operator in $[L^2(\mathbb{R}^3)]^2$; its precise definition will be given in section 2.

It is well known that there exist magnetic fields $\boldsymbol{B}$ for which $\mathbb{P}_A$ has zero modes, i.e. eigenvectors corresponding to an eigenvalue at 0 (see [2]). An important consequence is that there cannot be Sobolev and Hardy-type inequalities associated with such magnetic fields. Another important implication is that for such magnetic fields there cannot be an analogue of the Cwikel–Lieb–Rosenblum (CLR) inequality for the number of negative eigenvalues of $\mathbb{P}_A + V$ in terms of some L^p norm of the scalar potential V; this follows since any small negative perturbation V would produce negative eigenvalues, irrespective of the size of this norm of V. However, in [1] it is proved that the existence of zero modes is a rather rare phenomenon: specifically, it is proved that, with nul denoting nullity,

(i) for $|\boldsymbol{B}| \in L^{3/2}(\mathbb{R}^3)$, $\mathrm{nul}\,\mathbb{P}_{tA} = 0$ except for a finite number of values of t in any compact subset of $[0, \infty)$,
(ii) $\{\boldsymbol{B} : \mathrm{nul}\,\mathbb{P}_A = 0,\ \mathrm{curl}\boldsymbol{A} = \boldsymbol{B} \text{ and } |\boldsymbol{B}| \in L^{3/2}(\mathbb{R}^3)\}$ contains an open dense subset of $[L^{3/2}(\mathbb{R}^3)]^3$.

In this Letter, we use the analysis in [1] to prove that

$$\mathbb{P}_A \geqslant \delta(B)\, S_A \tag{3}$$

where $\delta(B)$ is a measure of the distance from $\boldsymbol{B}$ to the set of fields which yield zero modes, and S_A is the magnetic Schrödinger operator. Consequences of (3) are Sobolev, Hardy and CLR inequalities for $\mathbb{P}_A$ for all magnetic fields $\boldsymbol{B}$ in $[L^{3/2}(\mathbb{R}^3]^3$ which lie in the sets described in (i) and (ii) above.

2. Preliminaries

The formal operator (1) can be written as

$$\mathbb{P}_A = S_A + \boldsymbol{\sigma}\cdot\boldsymbol{B} \qquad \boldsymbol{B} = \mathrm{curl}\boldsymbol{A} \tag{4}$$

where S_A is the formal magnetic Schrödinger operator

$$S_A = \left(\frac{1}{\mathrm{i}}\boldsymbol{\nabla} + \boldsymbol{A}\right)^2 \mathbb{I}_2 \equiv \sum_{j=1}^{3}\left(\frac{1}{\mathrm{i}}\partial_j + A_j\right)^2 \mathbb{I}_2 \tag{5}$$

where $\mathbb{I}_2$ is the 2×2 identity matrix and $\boldsymbol{\sigma}\cdot\boldsymbol{B}$ is the Zeeman term. A gauge transformation $\boldsymbol{A} \mapsto \boldsymbol{A} + \mathrm{d}f$ does not affect $\mathrm{nul}\,\mathbb{P}_A$ and so our results will be independent of gauge. We denote $[L^2(\mathbb{R}^3)]^2$ by $\mathcal{H}$, and its usual inner product and norm by

$$(f, g) = \int_{\mathbb{R}^3} f\cdot\bar{g}\,\mathrm{d}\boldsymbol{x} \qquad \|f\| = (f, f)^{1/2}$$

where $f\cdot\bar{g}$ is the $\mathbb{C}^2$-inner product. We shall assume throughout that

$$A_j \in L^2_{\mathrm{loc}}(\mathbb{R}^3) \qquad j = 1, 2, 3 \tag{6}$$

and

$$|\boldsymbol{B}| \in L^{3/2}(\mathbb{R}^3). \tag{7}$$

Also, the following facts from [1] will be needed:

- The operators $\mathbb{P}_A$, S_A are defined to be the Friedrichs extensions of (4), (5) respectively on $[C_0^\infty(\mathbb{R}^3)]^2$. They have the same form domain, namely $\mathcal{Q}(S_A)$, the completion of $[C_0^\infty(\mathbb{R}^3)]^2$ with respect to the norm

$$\|\varphi\|_{1,A} = \left\{\left\|\left(\frac{1}{\mathrm{i}}\boldsymbol{\nabla} + \boldsymbol{A}\right)\varphi\right\|^2 + \|\varphi\|^2\right\}^{1/2}. \tag{8}$$

- S_A and the form sum $\mathbb{P} := \mathbb{P}_A + |\boldsymbol{B}|$ have no zero modes, and so have dense domain and range in $\mathcal{H}$. Also, $\mathcal{D}(\mathbb{P}^{1/2}) = \mathcal{D}(S_A^{1/2}) = \mathcal{Q}(S_A)$.

- Let H^1_A, $\mathbb{H}^1_B$ be the completion of $\mathcal{Q}(S_A)$ with respect to the norms

$$\|\phi\|_{\mathrm{H}^1_A} := \|S_A^{1/2}\phi\| \tag{9}$$

$$\|\phi\|_{\mathbb{H}^1_B} := \|\mathbb{P}^{1/2}\phi\| \tag{10}$$

respectively. We may choose an $\boldsymbol{A}$ to satisfy div$\boldsymbol{A} = 0$ and $|\boldsymbol{A}| \in L^3(\mathbb{R}^3)$ (see [1, lemma 2.2]) and in this case we have the continuous embeddings

$$\mathbb{H}^1_B \hookrightarrow \mathrm{H}^1_A \hookrightarrow \mathrm{H}^1_0 \hookrightarrow \left[L^6(\mathbb{R}^3)\right]^2. \tag{11}$$

Here, H^1_0 is the space H^1_A with $A_j = 0,\ j = 1, 2, 3$; it is not a subspace of $\mathcal{H}$ but, on account of the Hardy inequality, can be identified with the functional space

$$\left\{u \in \left[\mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^3)\right]^2 \ :\ \|u\|^2_{\mathrm{H}^1_0} + \left\|\frac{u}{|\cdot|}\right\|^2 < \infty\right\} \tag{12}$$

and $\|\cdot\|_{\mathrm{H}^1_0}$ is equivalent to the norm

$$\left\{\|u\|^2_{\mathrm{H}^1_0} + \left\|\frac{u}{|\cdot|}\right\|^2\right\}^{1/2}. \tag{13}$$

- The map $\mathbb{P}^{-1/2}$ extends to a unitary map

$$U\ :\ \mathcal{H} \longrightarrow \mathbb{H}^1_B \qquad U = \mathbb{P}^{-1/2} \ \text{ on } \ \mathrm{Range}(\mathbb{P}^{1/2}) \tag{14}$$

and

$$\mathcal{S} \ :=\ |\boldsymbol{B}|^{1/2}U\ :\ \mathcal{H} \longrightarrow \mathcal{H}$$

is continuous, where $f \mapsto |\boldsymbol{B}|^{1/2}f\ :\ \mathbb{H}^1_B \longrightarrow \mathcal{H}$ is continuous. In fact, $\mathcal{S}\mathcal{S}^*$ is compact.
- And finally

$$\mathbb{P}_A u = 0 \ \Leftrightarrow\ F(U^{-1}u) = 0 \tag{15}$$

where $F = 1 - \mathcal{S}\mathcal{S}^*$. Note that in (15) it is understood that

$$u \in \mathcal{D}(\mathbb{P}_A) \subset \mathbb{H}^1_B.$$

Thus nul $\mathbb{P}_A \leqslant$ nul F, with equality if and only if

$$F\phi = 0 \ \Rightarrow\ U\phi \in \mathcal{H} \cap \mathbb{H}^1_B. \tag{16}$$

3. The main result

Theorem. *Suppose that $\boldsymbol{B}$ is such that $F = 1 - \mathcal{S}\mathcal{S}^*$ has no zero mode, and set*

$$\delta(B) \ :=\ \inf_{\|f\|=1,\,Uf\in\mathcal{H}\cap\mathbb{H}^1_B} \left\|[1 - \mathcal{S}^*\mathcal{S}]f\right\|^2. \tag{17}$$

Then $\delta(B) > 0$ and

$$\mathbb{P}_A \ \geqslant\ \delta(B)S_A. \tag{18}$$

It follows that:

(i) (Sobolev embedding) for all $\phi \in \mathcal{D}(\mathbb{P}_A^{1/2}) = \mathcal{Q}(S_A)$,

$$\left\|\mathbb{P}_A^{1/2}\phi\right\|^2 \ \geqslant\ \frac{\delta(B)}{\gamma}\left\|\phi\right\|^2_{[L^6(\mathbb{R}^3)]^2} \tag{19}$$

where γ is the norm of the embedding $\mathrm{H}^1_0 \hookrightarrow \left[L^6(\mathbb{R}^3)\right]^2$;

(ii) *(Hardy inequality) for all* $\phi \in \mathcal{Q}(S_A)$,

$$\left\|\mathbb{P}_A^{1/2}\phi\right\|^2 \geqslant \frac{\delta(B)}{4}\left\|\frac{\phi}{|\cdot|}\right\|^2; \tag{20}$$

(iii) *(CLR inequality) for* $V_- \in L^{3/2}(\mathbb{R}^3)$*, the number* $N(\mathbb{P}_A + V)$ *of negative eigenvalues* $-\lambda_n$ *of* $\mathbb{P}_A + V$ *satisfies*

$$N(\mathbb{P}_A + V) \leqslant c[\delta(B)]^{-3/2}\int_{\mathbb{R}^3} V_-^{3/2}\,\mathrm{d}\boldsymbol{x} \tag{21}$$

where c is the best constant in the CLR inequality for S_A*. A consequence of (21) is that*

$$\sum \lambda_n^{\nu} \leqslant c[\delta(B)]^{-3/2}\int_{\mathbb{R}^3} V_-^{\nu+3/2}\,\mathrm{d}\boldsymbol{x} \tag{22}$$

for any $\nu \geqslant 0$.

Proof. If F has no zero mode, the compact operator $\mathcal{S}\mathcal{S}^*$ on $\mathcal{H}$ does not have eigenvalue 1 and hence neither does $\mathcal{S}^*\mathcal{S}$, since

$$\sigma(\mathcal{S}\mathcal{S}^*)\setminus\{0\} = \sigma(\mathcal{S}^*\mathcal{S})\setminus\{0\}.$$

Hence $\delta(B) > 0$, and for any $f \in \mathcal{H}$ with $Uf \in \mathcal{H}\cap\mathbb{H}_B^1$

$$\begin{aligned}\delta(B)\|f\|^2 &\leqslant \|(1-\mathcal{S}^*\mathcal{S})f\|^2\\ &= \|f\|^2 - 2(\mathcal{S}^*\mathcal{S}f, f) + \|\mathcal{S}^*\mathcal{S}f\|^2.\end{aligned}$$

Let $f = \mathbb{P}^{1/2}\phi$. Then $Uf = \phi$ and $\mathcal{S}f = |\boldsymbol{B}|^{1/2}\phi$ from (14), and so

$$\begin{aligned}\delta(B)\|\mathbb{P}^{1/2}\phi\|^2 &\leqslant \|\mathbb{P}^{1/2}\phi\|^2 - 2\||\boldsymbol{B}|^{1/2}\phi\|^2 + \|\mathcal{S}^*|\boldsymbol{B}|^{1/2}\phi\|^2\\ &= \|\mathbb{P}_A^{1/2}\phi\|^2 - \||\boldsymbol{B}|^{1/2}\phi\|^2 + \|\mathcal{S}^*|\boldsymbol{B}|^{1/2}\phi\|^2\end{aligned} \tag{23}$$

since $\mathbb{P} = \mathbb{P}_A + |\boldsymbol{B}|$ in the form sense. We also have for any $h \in \mathrm{Range}(\mathbb{P}^{1/2})$

$$\|\mathcal{S}h\| = \||\boldsymbol{B}|^{1/2}\mathbb{P}^{-1/2}h\| \leqslant \|h\|$$

since $\mathbb{P} \geqslant |\boldsymbol{B}|$, and this implies $\|\mathcal{S}\| \leqslant 1$ in view of the range of $\mathbb{P}^{1/2}$ being dense in $\mathcal{H}$. Thus $\|\mathcal{S}^*\| = \|\mathcal{S}\| \leqslant 1$ and from (23)

$$\delta(B)\|\mathbb{P}^{1/2}\phi\|^2 \leqslant \|\mathbb{P}_A^{1/2}\phi\|^2$$

whence $\mathbb{P}_A \geqslant \delta(B)\mathbb{P} \geqslant \delta(B)S_A$. The conclusions (i)–(iii) are consequences of these inequalities for S_A. □

Remark 1. If any one of the inequalities (19)–(21) is satisfied, $\mathbb{P}_A$ has no zero modes. Whether or not $\mathrm{nul}\,\mathbb{P}_A = 0$ implies that $\delta(B) > 0$ is not clear. Note that the infimum in (17) is taken over the subspace of $\mathcal{H}$ in which $\mathbb{P}_A$ and F have common nullity.

Remark 2. Let $\mathcal{S}$ in (14) be denoted by $\mathcal{S}_B$ and $F_B = 1 - \mathcal{S}_B\mathcal{S}_B^*$. The results in [1] which yield (i) and (ii) in section 1 are:

(iii) for $|\boldsymbol{B}| \in L^{3/2}(\mathbb{R}^3)$, $\mathrm{nul}\,F_{t B} = 0$ except for a finite number of values of t in any compact subset of $[0, \infty)$,

(iv) $\{\boldsymbol{B} : \mathrm{nul}\,F_B = 0,\ |\boldsymbol{B}| \in L^{3/2}(\mathbb{R}^3)\}$ is an open dense subset of $[L^{3/2}(\mathbb{R}^3)]^3$.

Thus (18)–(22) represent the typical situation, and fail to hold only for exceptional magnetic fields $\boldsymbol{B}$.

It follows from (11) (see [1, lemma 3.1]) that $\|\mathcal{S}\|^2 \leqslant \gamma_0^2 \||\boldsymbol{B}|\|_{L^{3/2}(\mathbb{R}^3)}$, where γ_0 is the norm of $\mathbb{H}_B^1 \hookrightarrow [L^6(\mathbb{R}^3)]^2$. Hence, if $\gamma_0^2 \|\boldsymbol{B}\|_{L^{3/2}(\mathbb{R}^3} < 1$,

$$\delta(B) \geqslant 1 - \gamma_0^2 \|\boldsymbol{B}\|_{L^{3/2}(\mathbb{R}^3)}$$

and from (21)

$$N(\mathbb{P}_A + V) \leqslant c[1 - \gamma_0^2 \|\boldsymbol{B}\|_{L^{3/2}(\mathbb{R}^3}]^{-3/2} \|V_-\|^{3/2}_{L^{3/2}(\mathbb{R}^3}. \quad (24)$$

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